MATH 303 – Measures and Integration
Homework 2

Problem 1. Suppose $\mathcal{B}$ is an infinite σ-algebra (on an infinite set X).

(a) Show that $\mathcal{B}$ contains an infinite sequence $(E_n)_{n\in\mathbb{N}}$ of pairwise disjoint sets.

(b) Deduce that $\mathcal{B}$ has at least the cardinality of the continuum.

Solution: (a) We will prove the following lemma to enable an inductive argument.

Lemma 1. *Let $\mathcal{C}$ be an infinite σ-algebra on an infinite set Y, and let $E \in \mathcal{C}$ with $\emptyset \neq E \neq Y$. Then either*

$$\mathcal{C}_E = \{C \cap E : C \in \mathcal{C}\}$$

is an infinite σ-algebra on E or

$$\mathcal{C}_{Y\setminus E} = \{C \setminus E : C \in \mathcal{C}\}$$

is an infinite σ-algebra on $Y \setminus E$.

Proof. First, let us check that $\mathcal{C}_E$ is a σ-algebra on E (applying the same argument to the set $Y \setminus E$ will also show that $\mathcal{C}_{Y\setminus E}$ is a σ-algebra on $Y \setminus E$). We may write $E = Y \cap E$, so $E \in \mathcal{C}_E$. If $B = C \cap E \in \mathcal{C}_E$, then $E \setminus B = E \cap (Y \setminus C) \in \mathcal{C}_E$, since $\mathcal{C}$ is closed under complementation. Given sets $B_n = C_n \cap E \in \mathcal{C}_E$, we may form their union $\bigcup_{n\in\mathbb{N}} B_n = (\bigcup_{n\in\mathbb{N}} C_n) \cap E \in \mathcal{C}_E$. Hence, $\mathcal{C}_E$ is a σ-algebra.

Now let us prove that one of the σ-algebras $\mathcal{C}_E$ or $\mathcal{C}_{Y\setminus E}$ is infinite. Suppose for contradiction that $\mathcal{C}_E$ and $\mathcal{C}_{Y\setminus E}$ are both finite. Then given any set $C \in \mathcal{C}$, we may write $C = (C\cap E)\cup(C\setminus E)$. But by assumption, there are only finitely many choices for $C \cap E$ and $C \setminus E$, so there are only finitely many such sets C. This is a contradiction, since $\mathcal{C}$ is an infinite σ-algebra. Thus, one of the σ-algebras $\mathcal{C}_E$ or $\mathcal{C}_{Y\setminus E}$ must be infinite. □

We will now use Lemma 1 to construct the sequence $(E_n)_{n\in\mathbb{N}}$ by induction. Let $A_1 \in \mathcal{B}$ be an arbitrary measurable set with $\emptyset \neq A_1 \neq X$. By Lemma 1, either $\mathcal{B}_{A_1}$ or $\mathcal{B}_{X\setminus A_1}$ is infinite. Let $E_1 \in \{A_1, X \setminus A_1\}$ such that $\mathcal{B}_{X\setminus E_1}$ is infinite. Write $X_1 = X \setminus E_1$ and $\mathcal{B}_1 = \mathcal{B}_{X_1}$.

Suppose we have chosen pairwise disjoint sets $E_1, \dots, E_n$ so that for $X_n = X\setminus(E_1\cup\cdots\cup E_n)$, the σ-algebra $\mathcal{B}_n = \mathcal{B}_{X_n}$ is infinite. We then choose $A_{n+1} \in \mathcal{B}_n$ arbitrarily with $\emptyset \neq A_{n+1} \neq X_n$ and let $E_{n+1} \in \{A_{n+1}, X_n \setminus A_{n+1}\}$ such that $\mathcal{B}_{X_n\setminus A_{n+1}}$ is infinite. Such a choice for E_{n+1} is possible by applying Lemma 1 with $Y = X_n$ and $\mathcal{C} = \mathcal{B}_n$. Put $X_{n+1} = X_n \setminus E_{n+1} = X \setminus (E_1 \cup \cdots \cup E_n \cup E_{n+1})$ and $\mathcal{B}_{n+1} = \mathcal{B}_{X_{n+1}}$. Note that E_{n+1} is disjoint from all of the sets $E_1, \dots, E_n$ since $E_{n+1} \subseteq X_n$.

Thus, by induction, we have constructed an infinite sequence of nonempty pairwise disjoint elements of $\mathcal{B}$.

(b) To show that $\mathcal{B}$ has at least the cardinality of the continuum, it suffices to produce an injective map from $\{0, 1\}^{\mathbb{N}}$ to $\mathcal{B}$. Let $(E_n)_{n\in\mathbb{N}}$ be the sequence from part (a). Given an infinite binary sequence $\omega \in \{0,1\}^{\mathbb{N}}$, define

$$E_\omega := \bigsqcup_{n:\omega(n)=1} E_n.$$

Since the sets $(E_n)_{n\in\mathbb{N}}$ are pairwise disjoint, we have

$$E_\omega \triangle E_{\omega'} = \bigsqcup_{n:\omega(n)\neq\omega'(n)} E_n$$

for $\omega, \omega' \in \{0,1\}^{\mathbb{N}}$. Thus, $\omega \mapsto E_\omega$ is an injective map and we are done.

Problem 2. Prove that the following sets are Borel sets in $\mathbb{R}$:

(a) The set of points of continuity

$$C_f = \{x \in \mathbb{R} : f \text{ is continuous at } x\}$$

for an arbitrary function $f : \mathbb{R} \to \mathbb{R}$.

(b) The set of points of convergence

$$\mathrm{Conv} = \{x \in \mathbb{R} : \lim_{n\to\infty} f_n(x) \text{ exists}\}$$

for an arbitrary sequence of continuous functions $f_n : \mathbb{R} \to \mathbb{R}$.

Solution: (a) Define the oscillation of f at a point $x \in \mathbb{R}$ by

$$\mathrm{osc}_f(x) = \lim_{\delta\to 0^+} \sup_{x-\delta<y_1,y_2<x+\delta} |f(y_1) - f(y_2)|.$$

Note that f is continuous at a point x if and only if $\mathrm{osc}_f(x) = 0$. We claim that the set $U_\varepsilon = \{x \in \mathbb{R} : \mathrm{osc}_f(x) < \varepsilon\}$ is open for every $\varepsilon > 0$. Suppose $x \in U_\varepsilon$. Then there exists $\delta_0 > 0$ such that if $\delta \in (0, \delta_0)$, then $\sup_{x-\delta<y_1,y_2<x+\delta} |f(y_1) - f(y_2)| < \varepsilon$. We will show $B\left(x, \frac{\delta_0}{2}\right) \subseteq U_\varepsilon$. Indeed, if $y \in B\left(x, \frac{\delta_0}{2}\right)$, then for any $\delta \in \left(0, \frac{\delta_0}{2}\right)$,

$$\sup_{y-\delta<y_1,y_2<y+\delta} |f(y_1) - f(y_2)| \leq \sup_{x-(\delta+\delta_0/2)<y_1,y_2<y+(\delta+\delta_0/2)} |f(y_1) - f(y_2)| < \varepsilon.$$

Thus, U_ε is open, so

$$C_f = \bigcap_{n\in\mathbb{N}} U_{1/n}$$

is a G_δ set.

(b) We will use the fact that $\lim_{n\to\infty} f_n(x)$ exists if and only if $(f_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence. Namely,

$$C = \bigcap_{n\in\mathbb{N}} \bigcup_{N\in\mathbb{N}} \bigcap_{m,k\geq N} \left\{x \in \mathbb{R} : |f_m(x) - f_k(x)| \leq \frac{1}{n}\right\}.$$

The innermost set is closed, since f_m and f_k are continuous functions for each $m, k \in \mathbb{N}$. Therefore, C is an $F_{\sigma\delta}$ set.

Problem 3. Let $(X, \mathcal{B})$ be a measurable space, and let $\mu : \mathcal{B} \to [0, \infty]$. Prove that μ is a measure if and only if it satisfies the following three properties:

- $\mu(\emptyset) = 0$;
- FINITE ADDITIVITY: for any disjoint sets $A, B \in \mathcal{B}$,
$$\mu(A \sqcup B) = \mu(A) + \mu(B);$$
- CONTINUITY FROM BELOW: if $E_1 \subseteq E_2 \subseteq \cdots \in \mathcal{B}$, then
$$\mu\left(\bigcup_{n\in\mathbb{N}} E_n\right) = \lim_{n\to\infty} \mu(E_n).$$

Solution: Countable additivity is stronger than finite additivity, and every measure is continuous from below by Proposition 2.15 from the lecture notes. Therefore, every measure satisfies $\mu(\emptyset) = 0$, finite additivity, and continuity from below.

Conversely, suppose $\mu(\emptyset) = 0$, and μ is finitely additive and continuous from below. We need to show that μ is countably additive. Let $(E_n)_{n\in\mathbb{N}}$ be a sequence of disjoint measurable sets. Let $E'_N = \bigsqcup_{n=1}^{N} E_n$. Then by finite additivity,

$$\mu(E'_N) = \sum_{n=1}^{N} \mu(E_n).$$

Moreover, $E'_1 \subseteq E'_2 \subseteq \dots$, so by continuity from below,

$$\mu\left(\bigsqcup_{n\in\mathbb{N}} E_n\right) = \mu\left(\bigcup_{N\in\mathbb{N}} E'_N\right) = \lim_{N\to\infty} \mu(E'_N) = \lim_{N\to\infty} \sum_{n=1}^{N} \mu(E_n) = \sum_{n=1}^{\infty} E_n.$$

That is, μ is countably additive, so μ is a measure as desired.